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2007 J. Phys. A: Math. Theor. 40 12619

(http://iopscience.iop.org/1751-8121/40/42/S08)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 12619–12627 doi:10.1088/1751-8113/40/42/S08

Constructing lump-like solutions of the Hirota–Miwa equation*

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Received 11 January 2007, in final form 6 March 2007
Published 2 October 2007
Online at stacks.iop.org/JPhysA/40/12619

Abstract

We construct special solutions of the Hirota–Miwa equation for which the τ-function is a polynomial in the independent variables. Three different methods are presented: direct construction (obtained also as a limit of the soliton solutions), and the derivation of the solutions in two different determinant forms, namely Grammian and Casorati. Introducing the appropriate ansatz, we write the Hirota–Miwa equation in a nonlinear form for a single variable. In terms of the latter, the solutions obtained are rational and are reminiscent of the lump solutions for the continuous analogue of the Hirota–Miwa equation, namely the KP equation.

PACS numbers: 02.30.Ik, 02.30.Jr

1. Introduction

Constructing explicit solutions of integrable evolution equations is particularly interesting both from a mathematical and a physical point of view. While the general solution is, in principle, obtained from a linear (usually integrodifferential) system, the explicit solutions allow one to form a clear mental picture of the dynamics of the evolution. In some cases, in fact, as in the case of solitons, these solutions incorporate the essential features of the dynamics. In other cases, as the special solutions of Painlevé equations, the singularity structure of the general solution reflects itself in the singularities of the solutions explicitly constructed. These facts explain the intense activity around the derivation of explicit solutions of integrable equations. It was through the observation of the existence of elastically interacting solitary waves that the KdV equation was first identified as a candidate (subsequently confirmed) for integrability.

* This work is dedicated to the memory of our mentor, Martin D Kruskal, zal.

The existence of N-soliton solutions for a given partial differential equation (PDE) is nowadays considered as an integrability criterion.

In this paper, we shall examine a family of special solutions of the discrete analogue of the KP equation. The familiar, continuous form of the latter is:

$$\partial_x(u_t + 6uu_x + u_{xxx}) + \sigma u_{yy} = 0 \tag{1.1}$$

where $\sigma = \pm 1$. Traditionally, the equation obtained with $\sigma = -1$ is called KP I, and the one for $\sigma = +1$, KPII. While the KPII equation possesses stable soliton solutions, KPI does not. On the other hand, the latter has localized solutions that decay algebraically as $x^2 + y^2 \to \infty$ and are called lumps. The lumps are part of a larger class of solutions which are rational functions of the independent variables. The particular character of the lumps resides in the fact that they are localized i.e., decrease at infinity and do not diverge. The lump solutions of KPI have been first obtained by Ablowitz and Satsuma [1]. Rational solutions, not included in the lump family were (almost simultaneously) obtained by Johnson and Thompson [2]. In a subsequent publication [3], Satsuma and Ablowitz gave the form of the N-lump solution of KPI. Localized structures in integrable evolution equations have also been studied by Fokas and Santini in [4]. In more recent studies [5], Ablowitz and collaborators have examined the lump solutions of KPI in an inverse scattering transform perspective. Dubrowsky in [6] has focussed on KPII and derived its rational, pole-like solutions.

While those studies have concentrated on the continuous KP equation, semi-continuous, differential-difference systems have also attracted attention over the years. Carstea [7] has studied the rational solutions of the discrete 1+1 Volterra equation while Villaroel, Chakravarty and Ablowitz [8] analysed the 2+1 Volterra system developing the inverse scattering transform and obtaining special solutions. Tam, Hu and Chian [9] have examined a host of (2+1)-dimensional lattices (in two continuous and one discrete variables) and derived their rational solutions. Curiously, the fully discrete analogue of the KP equations, which has been proposed independently by Hirota and Miwa (and which is deservedly known as the Hirota–Miwa (HM) equation) has not been studied from the point of view of the existence of rational solutions. These solutions will be the object of the present paper.

The HM equation is traditionally given in a bilinear form. One introduces the tau-function τ, which, as is well known, is often expressible as an entire function, and writes the HM equation as

$$(b-c)\tau_k\tau_{mn} + (c-a)\tau_m\tau_{nk} + (a-b)\tau_n\tau_{mk} = 0 \tag{1.2}$$

where $\tau_k \equiv \tau(k+1, m, n)$, $\tau_{mn} \equiv \tau(k, m+1, n+1)$ and similarly for the other indices. This particular gauge, where the sum of the coefficients is zero, is due to Hirota and ensures that a constant τ is a solution of (1.2). This constant solution represents the vacuum, i.e., the solution upon which all others will be built. We must stress here that this choice of coefficients is just one special gauge. It is indeed possible to put the three coefficients to any nonzero value without loss of generality. As a matter of fact, this goes even beyond the case where the coefficients are constant. As we have shown in [10–12] all integrable nonautonomous extensions of the HM equation can be gauge-reduced to an autonomous form with arbitrary coefficients.

In what follows, we shall construct the rational solutions of the HM equation, which in terms of τ are just polynomial solutions. We shall present the Grammian and Casorati determinant form of these solutions. Finally, introducing the adequate ansatz, we shall give a nonlinear form of the HM and present the solutions for the nonlinear variables.

2. A direct construction

Deriving rational solutions of the Hirota–Miwa equation is quite straightforward, provided one uses the adequate ansatz. Here we look for polynomial solutions, the first of which can be obtained starting from

$$\tau(n, m, k) = Ak + Bm + Cn + D. \tag{2.1}$$

Substituting (2.1) into (1.2) we find that the coefficients A, B, C must satisfy the dispersion relation

$$\frac{b-c}{A} + \frac{c-a}{B} + \frac{a-b}{C} = 0. \tag{2.2}$$

The higher solutions can be constructed along the same lines one follows for the derivation of the soliton solutions. We start by introducing the quantity $L_i = A_i k + B_i m + C_i n + D_i$. Solution (2.1) is thus simply $\tau^{(1)} = L_1$. Before proceeding further we can introduce a simplification which will make calculations more manageable. Since the Hirota–Miwa equation is homogeneous we can apply a global scaling of τ and put $C_i = 1$. There is no loss of generality here since $C_i = 0$ would entail $(a-b)A_i B_i = 0$ and if $a = b$ (1.2) would not be a genuine Hirota–Miwa equation, while $A_i B_i = 0$ in combination with $C_i = 0$ would lead to too poor a solution. Moreover, we can solve the dispersion relation for B_i and thus L_i contains just one effective parameter A_i.

With these simplifications, we proceed to the construction of the next polynomial solution. We find

$$\tau^{(2)} = L_1 L_2 + M_{12}. \tag{2.3}$$

Substituting into (1.2) we find that (2.3) is indeed a solution provided M_{12} satisfies the relation

$$M_{12} = \frac{A_1 A_2 (1 - A_1)(1 - A_2)}{(A_1 - A_2)^2}. \tag{2.4}$$

While we can proceed to the construction of the higher polynomial solutions in a straightforward way it is interesting at this stage to link this construction to that of the soliton solutions of Hirota–Miwa the existence of which is well established. Indeed we can start from the one-soliton solution $\tau = 1 + \theta$ where $\theta = \delta\alpha^k\beta^m\gamma^n$. The α, β, γ obey the dispersion relation $\mathcal{D}(\alpha, \beta, \gamma) = (b-c)(\beta\gamma + \alpha) + (c-a)(\alpha\gamma + \beta) + (a-b)(\alpha\beta + \gamma) = 0$. Putting $\alpha = 1 + \epsilon A, \beta = 1 + \epsilon B, \gamma = 1 + \epsilon C$ and taking the limit $\epsilon \to 0$ while choosing $\delta = -1 + \epsilon D$ we obtain for τ the first polynomial solution (2.1) while we can easily verify that the dispersion relation $\mathcal{D}(\alpha, \beta, \gamma) = 0$ goes over to (2.2). The quadratic solution (2.3) can be obtained from the two-soliton solution. We start from $\tau = 1 + \theta_1 + \theta_2 + \mu_{12}\theta_1\theta_2$ where μ_{12} is given by $\mu_{12} = -\alpha_2\beta_2\gamma_2\mathcal{D}(\alpha_1/\alpha_2, \beta_1/\beta_2, \gamma_1/\gamma_2)/\mathcal{D}(\alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2)$. We apply the same ansatz as above for $\alpha_i, \beta_i, \gamma_i$, take the limit $\epsilon \to 0$ and obtain precisely (2.3) where M_{12} is obtained from μ_{12} through $\mu_{12} = 1 + \epsilon^2 M_{12}$.

Just as in the case of soliton solutions, M_{12} will be the building block of the higher polynomial solutions of the Hirota–Miwa equation. We thus have

$$\tau^{(3)} = L_1 L_2 L_3 + M_{12} L_3 + M_{13} L_2 + M_{23} L_1 \tag{2.5}$$

at order three (with obvious expressions for M_{ij}). We remark that no new quantity enters at this order and everything is fixed by the solution at order two. This was expected since the three-soliton solution is expressed in terms of the μ_{ij} defined at order two, without the introduction of any new quantity. The limit $\epsilon \to 0$ preserves this property, and leads exactly to (2.5). In the same way we find, at order four, the expression

$$\begin{aligned}\tau^{(4)} = {} & L_1 L_2 L_3 L_4 + M_{12} L_3 L_4 + M_{13} L_2 L_4 + M_{23} L_1 L_4 + M_{14} L_2 L_3 + M_{24} L_1 L_3 \\ & + M_{34} L_1 L_2 + M_{12} M_{34} + M_{13} M_{24} + M_{14} M_{23}.\end{aligned} \tag{2.6}$$

This expression is again what one obtains from the 4-soliton solution in the limit $\epsilon \to 0$. Thus the polynomial solutions of the Hirota–Miwa equation can be constructed at any order in an algorithmic way by writing the homogeneous symmetrical combination of L_i, M_{ij} (with weight 2 for the M_{ij}).

3. The Grammian solutions

The polynomial solutions (2.1), (2.3), (2.5), (2.6) we constructed in the previous section, have a natural representation as Gramm-type determinants. It is well known [13, 14] that the Hirota–Miwa equation (1.2) possesses Grammian solutions, formulated in terms of so-called (vacuum) eigenfunctions φ and (vacuum) adjoint eigenfunctions φ^*, i.e. functions that satisfy the dispersion relations:

$$\Delta_k^- \varphi = \Delta_m^- \varphi = \Delta_n^- \varphi, \qquad \Delta_k \varphi^* = \Delta_m \varphi^* = \Delta_n \varphi^* \tag{3.1}$$

where the operators Δ^- and Δ represent resp. backward ($\Delta_k^- f(k,m,n) = a[f(k,m,n) - f(k-1,m,n)]$, etc) and forward difference operators (e.g. $\Delta_m f(k,m,n) = b[f(k,m+1,n) - f(k,m,n)]$, etc). In terms of N pairs of such functions we define N^2 Grammian elements $\Omega(\varphi_i, \varphi_j^*)$ $(i, j = 1, \dots, N)$ as the potentials that satisfy the following conditions: $\Delta_k^- \Omega(\varphi_i, \varphi_j^*) = \varphi_i(k,m,n)\varphi_j^*(k-1,m,n)$, $\Delta_m^- \Omega(\varphi_i, \varphi_j^*) = \varphi_i(k,m,n)\varphi_j^*(k,m-1,n)$, $\Delta_n^- \Omega(\varphi_i, \varphi_j^*) = \varphi_i(k,m,n)\varphi_j^*(k,m,n-1)$. This overdetermined set of equations is compatible because of (3.1). The determinant obtained from these Grammian elements

$$\tau = \begin{vmatrix} \Omega(\varphi_1, \varphi_1^*) & \Omega(\varphi_1, \varphi_2^*) & \cdots & \Omega(\varphi_1, \varphi_N^*) \\ \Omega(\varphi_2, \varphi_1^*) & \Omega(\varphi_2, \varphi_2^*) & \cdots & \Omega(\varphi_2, \varphi_N^*) \\ \vdots & \vdots & & \vdots \\ \Omega(\varphi_N, \varphi_1^*) & \Omega(\varphi_N, \varphi_2^*) & \cdots & \Omega(\varphi_N, \varphi_N^*) \end{vmatrix} \tag{3.2}$$

can then be shown to satisfy the Hirota–Miwa equation [13, 14].

To construct the polynomial solutions we obtained in the previous section, we choose the following eigenfunctions and adjoint eigenfunctions:

$$\varphi_i = \left(\frac{a}{\alpha_i}\right)^k \left(\frac{b}{\beta_i}\right)^m \left(\frac{c}{\gamma_i}\right)^n, \qquad \varphi_i^* = \frac{1}{\varphi_i} \qquad (i = 1, \dots, N) \tag{3.3}$$

where the parameters α_i, β_i and γ_i are required to satisfy the dispersion relations: $a - \alpha_i = b - \beta_i = c - \gamma_i$.

For these functions one immediately finds that $\Delta_k^- \Omega(\varphi_i, \varphi_i^*) = a/\alpha_i$, $\Delta_m^- \Omega(\varphi_i, \varphi_i^*) = b/\beta_i$ and $\Delta_n^- \Omega(\varphi_i, \varphi_i^*) = c/\gamma_i$, and hence the diagonal elements of the Grammian determinant $\Omega_{ii} = \Omega(\varphi_i, \varphi_i^*)$ are obviously given by:

$$\Omega_{ii} = \frac{k}{\alpha_i} + \frac{m}{\beta_i} + \frac{n}{\gamma_i} + \theta_i \qquad (i = 1, \dots, N) \tag{3.4}$$

where θ_i are arbitrary constants.

The off-diagonal elements $\Omega_{ij} = \Omega(\varphi_i, \varphi_j^*)$ $(i \neq j)$ are also easily calculated:

$$i \neq j: \qquad \Omega_{ij} = \omega_{ij} \left(\frac{\alpha_j}{\alpha_i}\right)^k \left(\frac{\beta_j}{\beta_i}\right)^m \left(\frac{\gamma_j}{\gamma_i}\right)^n + \theta_{ij} \tag{3.5}$$

with arbitrary constants θ_{ij} and with pre-factors ω_{ij} given by:

$$\omega_{ij} = \frac{1}{\alpha_j - \alpha_i} = \frac{1}{\beta_j - \beta_i} = \frac{1}{\gamma_j - \gamma_i} \tag{3.6}$$

(these last equalities being satisfied by virtue of the dispersion relations for the parameters $\alpha_i, \beta_i, \gamma_i$).

Hence, the Grammian determinant $\tau_N = |\Omega_{ij}|_{i,j=1,\ldots,N}$, with elements Ω_{ij} as given by (3.4) and (3.5) yields a solution for the HM equation if α_i, β_i and γ_i satisfy: $a - \alpha_i = b - \beta_i = c - \gamma_i$. These solutions are related to the polynomial ones presented in section 2 of the paper in the following way.

As one can always multiply τ in the HM equation by a constant without changing the fact that it is a solution, we define a new solution $\tau^{(N)}$ to (1.2) as: $\tau^{(N)} = \left(\prod_{i=1}^{N} \gamma_i\right)\tau_N$. We also define:

$$L_i = \gamma_i \Omega_{ii} \equiv A_i k + B_i m + n + D_i \tag{3.7}$$

with parameters

$$D_i = \gamma_i \theta_i, \qquad A_i = 1 + \frac{c-a}{\alpha_i}, \qquad \text{i.e.} \quad \alpha_i = \frac{a-c}{1-A_i} \tag{3.8}$$

and

$$B_i = \frac{(a-c)A_i}{(a-b)A_i + (b-c)} \tag{3.9}$$

taking into account the dispersion relations for $\alpha_i, \beta_i, \gamma_i$. Note that B_i as given by (3.9) immediately satisfy condition (2.2). It can also be easily verified from (3.6) that

$$\gamma_i \omega_{ij} = \frac{A_i(1-A_j)}{A_j - A_i}. \tag{3.10}$$

Hence, setting all off-diagonal constants $\theta_{ij} = 0$ in $\tau^{(N)}$, it can be seen that the determinant $|\Omega_{ij}|_{i,j=1,\ldots,N}$ is explicitly independent of the exponential functions that appear in (3.5) and, upon multiplication of every row by the appropriate factor (i.e., multiplying the ith row by γ_i), we find that

$$\tau^{(N)} = |T_{ij}|_{i,j=1,\ldots,N} \tag{3.11}$$

for

$$T_{ii} = L_i, \qquad T_{ij} = \frac{A_i(1-A_j)}{A_j - A_i} \qquad (i \neq j) \tag{3.12}$$

is a solution of the HM equation, for general size N. This Grammian determinant coincides exactly with the cases $N = 1, 2, 3, 4$ constructed by the direct method in section 2 of the paper.

4. The Casorati solutions

Solutions to the Hirota–Miwa equation often also permit a representation as Casorati-type determinants. In general one can show (see again [13, 14]) that the following determinant satisfies the Hirota–Miwa equation (1.2) :

$$\tau(\varphi_1, \ldots, \varphi_N) = \begin{vmatrix} \varphi_1 & \Delta_k^- \varphi_1 & (\Delta_k^-)^2 \varphi_1 & \cdots & (\Delta_k^-)^{N-1} \varphi_1 \\ \varphi_2 & \Delta_k^- \varphi_2 & (\Delta_k^-)^2 \varphi_2 & \cdots & (\Delta_k^-)^{N-1} \varphi_2 \\ \vdots & \vdots & \vdots & & \vdots \\ \varphi_N & \Delta_k^- \varphi_N & (\Delta_k^-)^2 \varphi_N & \cdots & (\Delta_k^-)^{N-1} \varphi_N \end{vmatrix}. \tag{4.1}$$

The φ_i $(i = 1, \ldots, N)$ are (vacuum) eigenfunctions, i.e. functions that satisfy the dispersion relations (3.1) for backward difference operators Δ^-. Note that due to these dispersion relations, the above definition in fact does not depend on the precise independent variable that is used to define the difference operators that appear in it: one could just as well use the variables m or n instead.

A fundamental property of the Hirota–Miwa equation is that it is gauge-invariant, namely that one can multiply τ by the exponential of an arbitrary linear function of k, m and n. This gauge invariance combined with the determinant form of τ allows us to multiply *each row* in the determinant by the exponential of an arbitrary linear function of k, m and n. We are thus led to adopt as the eigenfunctions φ_i that define the Casorati determinant, products of the L_i defined in (3.7) (i.e. with parameters B_i given by (3.9) in terms of A_i) with some appropriate gauge factors ψ_i and in which we explicitly write the arbitrary constant term in the polynomial as $D_i + \delta_i$ (for some extra constants δ_i which remain to be determined):

$$\phi_i = (L_i + \delta_i)\psi_i, \qquad \psi_i = \left(\frac{a-c}{a(1-A_i)}\right)^{-k}\left(\frac{(a-c)A_i}{bB_i(1-A_i)}\right)^{-m}\left(\frac{(a-c)A_i}{c(1-A_i)}\right)^{-n}. \tag{4.2}$$

It is easily verified that $\Delta_k^-(\phi_i) = \Delta_m^-(\phi_i) = \Delta_n^-(\phi_i) = \frac{c-aA_i}{1-A_i}\phi_i + \frac{(a-c)A_i}{1-A_i}\psi_i$ and hence, that ϕ_i are indeed eigenfunctions that will define a solution of type (4.1). In fact, one has that $(p = 0, 1, 2, \ldots)$

$$(\Delta_k^-)^p\phi_i = \left(\frac{c-aA_i}{1-A_i}\right)^p\left[\phi_i + p\frac{(a-c)A_i}{c-aA_i}\psi_i\right]. \tag{4.3}$$

For simplicity we introduce the following shorthand notation:

$$\rho_i = \frac{c-aA_i}{1-A_i}, \qquad \sigma_i = \frac{(a-c)A_i}{c-aA_i} \tag{4.4}$$

in terms of which we can write the (i, j)th element of the Casorati determinant $\tau(\phi_1, \ldots, \phi_N)$ (as defined in (4.1)) as

$$(\rho_i)^{j-1}[L_i + \delta_i + (j-1)\sigma_i]\psi_i. \tag{4.5}$$

Hence, taking advantage of the fundamental gauge invariance of the solutions of the Hirota–Miwa equation, we can gauge away the factors ψ_i that appear in the rows of $\tau(\phi_1, \ldots, \phi_N)$ and we thus obtain a Casorati determinant $\widetilde{\tau}(\phi_1, \ldots, \phi_N) = \tau(\phi_1, \ldots, \phi_N)\prod_{i=1}^N \psi_i^{-1}$ whose elements now only contain polynomial expressions in k, m and n. Unfortunately, this determinant is—at least at first sight—not of the form (3.11), (3.12). It is however not difficult to see that by taking appropriate linear combinations of the columns, this determinant can be re-cast into the required form (i.e. into a form where L_i's only appear on the diagonal, the off-diagonal elements being all constant) if δ_i take some very precise (and unique) values. As there are only N of them, these particular values of δ_i are in fact already fully determined if one requires that the Casorati determinant $\widetilde{\tau}(\phi_1, \ldots, \phi_N)$ does not contain any terms of the form $\prod_{i=1,i\neq k}^N L_i$ (for $k = 1, \ldots, N$). In fact, the coefficient of such a term (for general k) can be expressed in terms of the Vandermonde determinant $V(\rho_1, \ldots, \rho_n) = \prod_{i,j=1,j>i}^N(\rho_j - \rho_i)$:

$$\delta_k V(\rho_1, \ldots, \rho_n) + \sigma_k\rho_k\partial_{\rho_k}V(\rho_1, \ldots, \rho_n). \tag{4.6}$$

Hence, in order for these coefficients to be zero, one finds that the constants δ_i have to take the values

$$\delta_i = \sigma_i\rho_i\sum_{j=1,j\neq i}^N \frac{1}{\rho_j - \rho_i} \qquad (i = 1, \ldots, N). \tag{4.7}$$

Using the explicit forms of ρ_i and σ_i (4.4), one finds that $\rho_j - \rho_i = \frac{(a-c)(A_i-A_j)}{(1-A_i)(1-A_j)}$ and hence that

$$\delta_i = A_i\sum_{j=1,j\neq i}^N \frac{1-A_j}{A_i - A_j} \qquad (i = 1, \ldots, N). \tag{4.8}$$

For example, at $N = 2$ this yields $\delta_1 = A_1 \frac{1-A_2}{A_1-A_2}$, $\delta_2 = A_2 \frac{1-A_1}{A_2-A_1}$, for which one finds that

$$\widetilde{\tau}(\phi_1, \phi_2) = (\rho_2 - \rho_1) \left[L_1 L_2 + \frac{\rho_1 \rho_2 \sigma_1 \sigma_2}{(\rho_1 - \rho_2)^2} \right] = (\rho_2 - \rho_1)[L_1 L_2 + M_{12}] \tag{4.9}$$

with M_{12} as in (2.4); i.e. up to a constant multiple one obtains the function $\tau^{(2)}$ as given in (2.3).

Similarly, for $N = 3$ one finds that $\delta_1 = A_1 \big[\frac{1-A_2}{A_1-A_2} + \frac{1-A_3}{A_1-A_3} \big]$, $\delta_2 = A_2 \big[\frac{1-A_1}{A_2-A_1} + \frac{1-A_3}{A_2-A_3} \big]$, $\delta_3 = A_3 \big[\frac{1-A_1}{A_3-A_1} + \frac{1-A_2}{A_3-A_2} \big]$ and accordingly that

$$\widetilde{\tau}(\phi_1, \phi_2, \phi_3) = V(\rho_1, \rho_2, \rho_3) \left[L_1 L_2 L_3 + \frac{\rho_2 \rho_3 \sigma_2 \sigma_3}{(\rho_2 - \rho_3)^2} L_1 + \frac{\rho_1 \rho_3 \sigma_1 \sigma_3}{(\rho_1 - \rho_3)^2} L_2 + \frac{\rho_1 \rho_2 \sigma_1 \sigma_2}{(\rho_1 - \rho_2)^2} L_3 \right] \tag{4.10}$$

which is a constant multiple of $\tau^{(3)}$ as given in (2.5).

In general one finds that $\tau^{(N)}$ as given by the Grammian determinant (3.11), is actually (gauge-) equivalent to a Casorati-type solution to the Hirota–Miwa equation:

$$\tau^{(N)} = V(\rho_1, \ldots, \rho_N)^{-1} \begin{vmatrix} L_1 + \delta_1 & (L_1 + \delta_1 + \sigma_1)\rho_1 & \cdots & (L_1 + \delta_1 + (N-1)\sigma_1)\rho_1^{N-1} \\ L_2 + \delta_2 & (L_2 + \delta_2 + \sigma_2)\rho_2 & \cdots & (L_2 + \delta_2 + (N-1)\sigma_2)\rho_2^{N-1} \\ \vdots & \vdots & & \vdots \\ L_N + \delta_N & (L_N + \delta_N + \sigma_N)\rho_N & \cdots & (L_N + \delta_N + (N-1)\sigma_N)\rho_N^{N-1} \end{vmatrix} \tag{4.11}$$

iff δ_i take values (4.8).

5. The nonlinear form of the Hirota–Miwa equation

While the construction of the solutions of the Hirota–Miwa equation in terms of τ is most fundamental, it is also interesting to obtain these solutions for the nonlinear form in which the Hirota–Miwa equation can be cast. Starting from (1.2) we multiply by τ and divide by the product $\tau_k \tau_m \tau_n$. We obtain thus the equation

$$(b - c)\frac{\tau \tau_{mn}}{\tau_m \tau_n} + (c - a)\frac{\tau \tau_{kn}}{\tau_k \tau_n} + (a - b)\frac{\tau \tau_{km}}{\tau_k \tau_m} = 0. \tag{5.1}$$

Next we introduce the three nonlinear variables

$$x = (b - c)\frac{\tau \tau_{mn}}{\tau_m \tau_n}, \qquad y = (c - a)\frac{\tau \tau_{kn}}{\tau_k \tau_n}, \qquad z = (a - b)\frac{\tau \tau_{km}}{\tau_k \tau_m}. \tag{5.2}$$

Given the definition of x, y, z, we can easily show that

$$\frac{x_k}{x} = \frac{y_m}{y} = \frac{z_n}{z}. \tag{5.3}$$

Moreover, from (5.1) we find readily

$$x + y + z = 0. \tag{5.4}$$

It is remarkable that this nonlinear form of the Hirota–Miwa equation does not contain the parameters a, b, c that appear in the bilinear form. As we have explained in the introduction, it is possible with the adequate gauge to bring the coefficients a, b, c to any value, but such a gauge will leave the nonlinear system invariant. Still, as these gauge transformations do not map the vacuum solutions for different gauges of the Hirota–Miwa equation into each other, the solutions of the nonlinear equation possess some extra (internal) parametric freedom

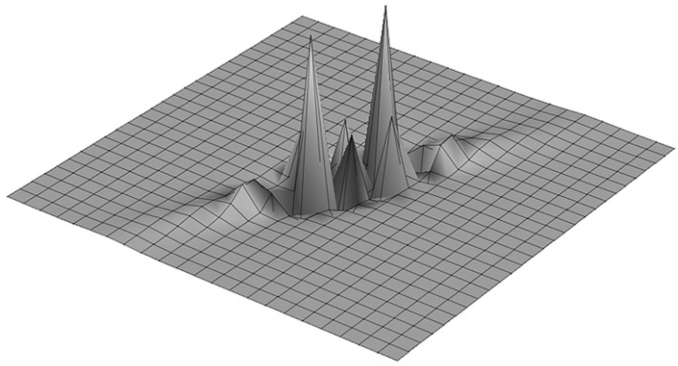

Figure 1. Plot of the function $x(0, m, n)$ corresponding to $\tau^{(2)}$ with parameter values $a = 1$, $b = 3, c = 2, A_1 = i, A_2 = A_1^* = -i, D_1 = D_2 = 0$. The range of the plot is: $(m, n) \in [-13, 13] \times [-11, 11]$.

due to the explicit dependence of the τ functions on the specific gauge a, b, c they were constructed for.

As a matter of fact, we can eliminate two of the variables and obtain the nonlinear expression of the Hirota–Miwa equation in terms of, say, x:

$$\begin{aligned} x_{km}x_{kmm}x_{kmn}x_{nn} &- x_{km}x_{kmm}x_{knn}x_{mn} - x_{kmm}x_{kmn}x_{mm}x_{nn} - x_{kmm}x_{kn}x_{knn}x_{mn} \\ &+ x_{kmm}x_{knn}x_{mm}x_{mn} + x_{kmm}x_{knn}x_{mn}^2 + x_{kmm}x_{knn}x_{mn}x_{nn} - x_{kmm}x_{mn}x_{mnn}x_{nn} \\ &- x_{kmn}^2x_{mm}x_{nn} + x_{kmn}x_{kn}x_{knn}x_{mm} - x_{kmn}x_{knn}x_{mm}x_{nn} + x_{kmn}x_{mm}x_{mmn}x_{nn} \\ &+ x_{kmn}x_{mm}x_{mnn}x_{nn} - x_{knn}x_{mm}x_{mmn}x_{mn} = 0. \end{aligned} \tag{5.5}$$

In order to illustrate the behaviour of the solutions of (5.5) we present in figure 1 below such a solution for x obtained with a quadratic τ given by equation (2.3) and with parameters such that $L_2 = L_1^*$.

As we can assess from this plot the solution is well localized in space and decreases as an inverse square as m or n go to infinity (after subtraction of the vacuum $x = 1$). Taking $L_2 = L_1^*$ ensures that the first term in (2.3) is positive but has as a consequence that M_{12} (from (2.4)) is negative. Thus τ may have a zero which would in principle lead to a diverging x. In that sense, these solutions are analogous to the singular solutions of KPII which is indeed the continuous limit of HM. Still, since we are looking for a solution on a lattice it is possible (and figure 1 is an example of this) to choose the parameters so as not to have any root of τ on the lattice points and guarantee the finiteness of x on the lattice by imposing that M_{12} and $L_1L_2 = |L_1|^2$ be irrationally related for all integers k, m, n.

6. Conclusion

In this paper we have investigated a particular class of solutions of the Hirota–Miwa (discrete KP) equation. These solutions, where the τ-function is polynomial, are the analogues of the ones obtained in the continuous case and which give rise to the lump solutions for KPI. Despite the fact that the straightforward continuous limit of HM leads to KPII (which does not possess lumps) our solutions may be constrained to be finite (and thus really lump-like) on the lattice points (but there is no way to prevent singularities at the continuous limit).

Three different approaches were presented for the construction of the solutions. The first was a direct method which can also be viewed as a construction of the polynomial solution by a special limit of the soliton one. The second and third methods consisted in representing the solution as a determinant, Grammian and Casorati respectively. We complemented our analysis by the derivation of the Hirota–Miwa in nonlinear form (with the adequate ansatz) and gave a graphical representation of a lump-like solution.

The methods we presented in this paper could be extended to other multidimensional lattice equations, which will be the object of some future work of ours.

Acknowledgments

V Papageorgiou acknowledges partial support of the programme 'C Caratheodory' of the Research Committee of the University of Patras and the hospitality of the Laboratoire IMNC of Paris VII-Paris XI Universities during his visit last year when this work was initiated.

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